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NOTE ON TWO TRANSFORMS OF PLANE CURVES AND THEIR FUNDAMENTAL GROUPS

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§1. Introduction. Let C be a projective curve and let $C^a = \{f(x,y) = 0\} \subset \mathbf{C}^2$ be the corresponding affine plane curve with respect to the affine coordinate space $\mathbf{C}^2 = \mathbf{P}^2 - \{Z = 0\}$, $x = X/Z$, $y = Y/Z$ and $f(x,y) = F(x,y,1)$.

In this note, we study two basic operations. For the detail, see [O7]. First we consider an n-fold cyclic covering $\varphi_n : \mathbf{C}^2 \to \mathbf{C}^2$, $\varphi_n(x,y) = (x, (y-\beta)^n + \beta)$, branched along a line $D = \{y = \beta\}$ for an arbitrary positive integer $n \geq 2$. Let $\mathcal{C}_n(C; D)$ be the projective closure of the pull back $\varphi_n^{-1}(C^a)$ of C^a. The behavior of φ_n at infinity gives an interesting effect on the fundamental group. In our previous paper [O6], we have studied the double covering φ_2 to construct some interesting plane curves, such as a Zariski's three cuspidal quartic and a conical six cuspidal sextic. Secondly we consider the following Jung transform of degree n, $J_n : \mathbf{C}^2 \to \mathbf{C}^2$, $J_n(x,y) = (x + y^n, y)$ and let $\mathcal{J}_n(C; L_\infty)$ be the projective compactification of $J_n^{-1}(C^a)$. Though J_n is an automorphism of $\mathbf{C}^2$, the behavior of J_n or $\mathcal{J}_n(C)$ at infinity is quite interesting. Both of φ_n and J_n can be extended canonically to rational mapping from $\mathbf{P}^2$ to $\mathbf{P}^2$ and they are not defined only at $[1;0;0]$ and constant along the line at infinity $L_\infty = \{Z = 0\}$. They have also the following similarity. For a generic φ_n and a generic J_n, there exist surjective homomorphisms

$$\Phi_n : \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C)) \to \pi_1(\mathbf{P}^2 - C), \quad \Psi_n : \pi_1(\mathbf{P}^2 - \mathcal{J}_n(C)) \to \pi_1(\mathbf{P}^2 - C)$$

and both kernels $\operatorname{Ker} \Phi_n$ and $\operatorname{Ker} \Psi_n$ are cyclic group of order n which are subgroups of the respective centers of $\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))$ and $\pi_1(\mathbf{P}^2 - \mathcal{J}_n(C))$ (Theorem (3.7) and Theorem (4.7)).

Both operations are useful to construct examples of interesting plane curves, starting from a simple plane curve. Applying this operation to a Zariski's three cuspidal quartic Z_4, we obtain new examples of plane curves $\mathcal{C}_n(Z_4)$ and $\mathcal{J}_n(Z_4)$ of degree $4n$ whose complement in $\mathbf{P}^2$ has a non-commutative finite fundamental group of order $12n$ (§5). We will construct a new example of Zariski pair $\{\mathcal{C}_3(Z_4), C_2\}$ of curves of degree 12 (§5).

§2. Basic properties of $\pi_1(\mathbf{P}^2 - \mathbf{C})$ and Zariski's pencil method. Let C be a reduced projective curve of degree d and let $C_1, \dots, C_r$ be the irreducible components of C and let d_i be the degree of C_i. So $d = d_1 + \cdots + d_r$. First we recall that the first homology of the complement is given by the Lefschetz duality and by the exact sequence of the pair $(\mathbf{P}^2, C)$ as follows.

$$H_1(\mathbf{P}^2 - C) \cong \mathbf{Z}^r/(d_1, \dots, d_r) \cong \mathbf{Z}^{r-1} \oplus \mathbf{Z}/d_0\mathbf{Z} \tag{2.1}$$

where $d_0 = \gcd(d_1, \dots, d_r)$ and $\mathbf{Z}^r = \mathbf{Z} \oplus \cdots \oplus \mathbf{Z}$ (r factors). In particular, if C is irreducible ($r = 1$), we have $H_1(\mathbf{P}^2 - C) \cong \mathbf{Z}/d\mathbf{Z}$ and $H_1(\mathbf{C}^2 - C^a) \cong \mathbf{Z}$ where $\mathbf{C}^2 := \mathbf{P}^2 - L_\infty$ and $C^a := C \cap L_\infty$.

(A) van Kampen-Zariski's pencil method. We fix a point $B_0 \in \mathbf{P}^2$ and we consider the pencil of lines $\{L_\eta,\ \eta \in \mathbf{P}^1\}$ through B_0. Taking a linear change of coordinates if necessary, we may assume that L_η is defined by $L_\eta = \{X - \eta Z = 0\}$ and $B_0 = [0;1;0]$ in homogeneous coordinates. Take $L_\infty = \{Z = 0\}$ as the line at infinity and we write $\mathbf{C}^2 = \mathbf{P}^2 - L_\infty$. Note that $L_\infty = \lim_{\eta \to \infty} L_\eta$. We assume that $L_\infty \not\subset C$. We consider the affine coordinates $(x,y) = (X/Z, Y/Z)$ on $\mathbf{C}^2$ and let $F(X,Y,Z)$ be the defining homogeneous polynomial of C and let $f(x,y) := F(x,y,1)$ be the affine equation of C. In this affine coordinates, the pencil line L_η is simply defined by $\{x = \eta\}$. As we consider two fundamental groups $\pi_1(\mathbf{P}^2 - C)$ and $\pi_1(\mathbf{P}^2 - C \cup L_\infty)$ simultaneously, we use the notations : $C^a = C \cap \mathbf{C}^2$ and $L_\eta^a = L_\eta \cap \mathbf{C}^2 \cong \mathbf{C}$. We identify hereafter L_η and L_η^a with $\mathbf{P}^1$ and $\mathbf{C}$ respectively by $y : L_\eta \cong \mathbf{P}^1$ for $\eta \neq \infty$. Note that the base point of the pencil B_0 corresponds to $\infty \in \mathbf{P}^1$.

We say that the pencil $\{L_\eta = \{x = \eta\},\ \eta \in \mathbf{C}\}$, is *admissible* if there exists an integer $d' \leq d$ which is independent of $\eta \in \mathbf{C}$ such that $C^a \cap L_\eta^a$ consists of d' points counting the multiplicity. This is equivalent to : *$f(x,y)$ has degree d' in y and the coefficient of $y^{d'}$ is a non-zero constant.* Note that if $B_0 \notin C$, L_η is admissible and $d' = d$. If $d' < d$, $B_0 \in C$ and the intersection multiplicity $I(C, L_\infty; B_0) = d - d'$.

Hereafter we assume that the pencil $\{L_\eta\}$ is admissible. A line L is called *generic* with respect to C if $C \cap L$ consists of d distinct points. A pencil line L_η is called *non-generic* with respect to C if L_η passes through a singular point of C^a or L_η is tangent to C^a. Otherwise L_η is called *generic.* Here we note that a generic pencil line L_{η_0} may not be generic as a line in $\mathbf{P}^2$ if $B_0 \in C$ and $d - d' \geq 2$ but L_{η_0} intersects transversely with C^a at d' points.

Let $\mathbf{C}_B$ be the line of the parameters of the pencil ($\mathbf{C}_B \cong \mathbf{C}$) and $\Sigma := \{\eta_1, \dots, \eta_\ell\}$ be parameters in $\mathbf{C}_B$ which corresponds to non-generic pencil lines. We fix a generic pencil line L_{η_0} and put $L_{\eta_0}^a \cap C^a = \{Q_1, \dots, Q_{d'}\}$. The complement $L_{\eta_0}^a - L_{\eta_0}^a \cap C^a$ is topologically $\mathbf{C}$ minus d'-points . We take a base point $b_0 \in L_{\eta_0}^a$ on the imaginary axis which is sufficiently near to B_0 and $b_0 \neq B_0$. We take a large disk Δ_{η_0} in the generic pencil line $L_{\eta_0}^a$ such that $\Delta_{\eta_0} \supset C \cap L_{\eta_0}^a$ and $b_0 \notin \Delta_{\eta_0}$. We orient the boundary of Δ_{η_0} counter-clockwise and let $\Omega = \partial \Delta_{\eta_0}$. We join Ω to the base point by a path L connecting b_0 and Ω along the imaginary axis. Let ω be the class of this loop $L \circ \Omega \circ L^{-1}$ in $\pi_1(L_{\eta_0}^a - L_{\eta_0}^a \cap C; b_0)$. We take free generators $g_1, \dots, g_{d'}$ of $\pi_1(L_{\eta_0}^a - L_{\eta_0}^a \cap C; b_0)$ so that g_i goes around Q_i counter-clockwise along a small circle and

$$\omega = g_{d'} \cdots g_1 \tag{2.2}$$

Put $G = \pi_1(L_{\eta_0}^a - L_{\eta_0}^a \cap C^a; b_0)$. Note that G is a free group of rank d' with generators $g_1, \dots, g_{d'}$. The fundamental group $\pi_1(\mathbf{C}_B - \Sigma; \eta_0)$ acts on G which we refer by *the monodromy action* of $\pi_1(\mathbf{C}_B - \Sigma; \eta_0)$. We recall this action quickly.

Take a large disk $\Delta \subset \mathbf{C}_B$ on the base space so that $\Delta \supset \Sigma$ and $\eta_0 \in \Delta$. So we have $\pi_1(\mathbf{C}_B - \Sigma; \eta_0) \cong \pi_1(\Delta - \Sigma; \eta_0)$. We take a system of free generators $\sigma_1, \dots, \sigma_\ell$ of $\pi_1(\Delta - \Sigma; \eta_0)$ which are represented by smooth loops in Δ, so that the product $\sigma_\ell \cdots \sigma_1$ is homotopic to the counter-clockwise oriented boundary of Δ. We take a large disk of radius R, $B(R) := \{y \in \mathbf{C}; |y| \leq R\}$ so that $B(R) \supset \bigcup_{\eta \in \Delta} C^a \cap L_\eta$ under the identification $y : L_\eta^a \cong \mathbf{C}$. We may assume that $b_0 \in L_{\eta_0} - B(2R)$. Take $g \in \pi_1(L_{\eta_0}^a - C^a \cap L_{\eta_0}^a; b_0)$ and $\sigma \in \pi_1(\mathbf{C}_B - \Sigma; \eta_0)$. Represent them by smooth loops $\alpha : (I, \partial I) \to (L_{\eta_0}^a - L_{\eta_0}^a \cap C; b_0)$ and $\tau : (I, \partial I) \to (\Delta - \Sigma; \eta_0)$ and construct a one-parameter family of deffeomorphisms $h_\theta : (L_{\eta_0}, C \cap L_{\eta_0}) \to (L_{\tau(\theta)}, C \cap L_{\sigma(\theta)})$, $0 \leq \theta \leq 1$ such that the composition

$$\mathbf{C} \xrightarrow{y^{-1}} L_{\eta_0}^a \xrightarrow{h_\theta} L_{\tau(\theta)}^a \xrightarrow{y} \mathbf{C}$$

is identity on $\mathbf{C} - B(2R)$. The action of $\sigma \in \pi_1(\mathbf{C}_B - \Sigma; \eta_0)$ on $g \in G$ is defined by $(g, \sigma) \mapsto [h_{2\pi} \circ \alpha]$. We denote this class by g^σ. Note that $\omega^g = \omega$ for any $g \in \pi_1(\mathbf{C}_B - \Sigma; \eta_0)$. The normal subgroups of G which is normally generated by $\{g^{-1}g^{\sigma}\ ;\ g \in G, \sigma \in \pi_1(\mathbf{C}_B - \Sigma; \eta_0)\}$ is called *the group of the monodromy relations* and we denote it by $\mathcal{M}$. Let $\mathcal{M}(\sigma_i) = \{g_j^{\sigma_i} g_j^{-1}; j = 1, \dots, d\}$. Then the group of the monodromy relations $\mathcal{M}$ is the minimal normal subgroup of G generated by $\bigcup_{i=1}^{\ell} \mathcal{M}(\sigma_i)$. By the definition, we have the relation $R(\sigma_i)\ :\ g_j = g_j^{\sigma_i}$ in the quotient group $G/\mathcal{M}$. We call $R(\sigma_i)$ *the monodromy relation for* σ_i. The following is a reformulation of a theorem of van-Kampen ([K]) to an affine situation with an admissible pencil. Let $j : L_{\eta_0}^a - L_{\eta_0} \cap C^a \to \mathbf{C}^2 - C^a$ and $\iota : \mathbf{C}^2 - C^a \to \mathbf{P}^2 - C$ be the respective inclusions.

Proposition (2.3). (1) *The canonical homomorphism* $j_\sharp\ :\ \pi_1(L_{\eta_0}^a - L_{\eta_0}^a \cap C^a; b_0) \to \pi_1(\mathbf{C}^2 - C^a; b_0)$ *is surjective and the kernel* $\operatorname{Ker} j_\sharp$ *is equal to* $\mathcal{M}$ *and therefore* $\pi_1(\mathbf{C}^2 - C^a; b_0)$ *is isomorphic to the quotient group* $G/\mathcal{M}$.
(2) *The canonical homomorphism* $\iota_\sharp : \pi_1(\mathbf{C}^2 - C^a; b_0) \to \pi_1(\mathbf{P}^2 - C; b_0)$ *is surjective. If* $B_0 \notin C$ *(so* $d' = d$*), the kernel* $\operatorname{Ker} \iota_\sharp$ *is normally generated by* $\omega = g_d \cdots g_1$.

Assume further that $B_0 \notin C$ *and* L_∞ *is generic. Then*
(3) ([O3]) ω *is in the center of* $\pi_1(\mathbf{C}^2 - C^a)$. *Therefore* $\operatorname{Ker}(\iota_\sharp) = \langle \omega \rangle \cong \mathbf{Z}$.
(4) $\iota_\sharp$ *induces an isomorphism of the commutator groups:* $\iota_{\sharp\mathcal{D}} : \mathcal{D}(\pi_1(\mathbf{C}^2 - C^a)) \xrightarrow{\cong} \mathcal{D}(\pi_1(\mathbf{P}^2 - C))$ *and an exact sequence of first homologies:* $0 \to \langle \omega \rangle \cong \mathbf{Z} \to H_1(\mathbf{C}^2 - C) \to H_1(\mathbf{P}^2 - C) \to 0$.

We usually denote $G/\mathcal{M}$ as $\pi_1(\mathbf{C}^2 - C^a; b_0) = \langle g_1, \dots, g_d; R(\sigma_1), \dots, R(\sigma_\ell) \rangle$. We call $\pi_1(\mathbf{C}^2 - C^a)$ *the fundamental group of a generic affine complement of* C if L_∞ is generic. Note that if L_∞ is generic, $\pi_1(\mathbf{C}^2 - C^a)$ does not depend on the choice of a line at infinity L_∞.

(B) Bracelets and lassos. An element $\rho \in \pi_1(\mathbf{P}^2 - C; b_0)$ is called *a lasso* for C_i if it is represented by a loop $\mathcal{L} \circ \tau \circ \mathcal{L}^{-1}$ where τ is a counter-clockwise oriented boundary of a small normal disk $D_i(P)$ of C_i at a regular point $P \in C_i$ such that $D_i(P) \cap (C \cup L_\infty) = \{P\}$ and $\mathcal{L}$ is a path connecting b_0 and τ. We call τ *a bracelet* for C_i. It is easy to see that any two bracelets τ and τ' for the same irreducible component, say C_i, are free homotopic. Therefore *the homotopy class of a lasso for* C_i *(or* L_∞*) is unique up to a conjugation.* We say that the line at infinity L_∞ is *central* for C if there is a lasso ω for L_∞ which is in the center of $\pi_1(\mathbf{C}^2 - C^a) = \pi_1(\mathbf{P}^2 - C \cup L_\infty)$. If L_∞ is generic for C, L_∞ is central by Proposition (2.3) but the converse is not always true (see Corollary (3.3.1) and Theorem (4.3)).

Assume that L_∞ is central for C and take an admissible pencil $\{L_\eta, \eta \in \mathbf{C}\}$ with the base point $B_0 \notin C$. Then $d' = d$ and ω defined by (2.2) is in the center of $\pi_1(\mathbf{C}^2 - C^a; b_0)$ as ω^{-1} is a lasso for L_∞. Thus we can replace the homotopy deformation of ω by free homotopy deformation of Ω. This viewpoint is quite useful in the later sections.

Remark (2.4). Suppose that $B_0 \notin C$ and L_∞ is *not generic.* Take $\Delta = \{\eta \in \mathbf{C}_B; |\eta| \le R\} \subset \mathbf{C}_B$ as before and we may assume that $\eta_0 \in \partial\Delta$ and let $\sigma_\infty := \partial\Delta$. The monodromy relation $g_i^{-1} g_i^{\sigma_\infty}$ is contained in the group of monodromy relations $\mathcal{M}$. We can also consider the monodromy relation around $\eta = \infty$. For this purpose, we identify $L_\eta \cong \mathbf{P}^1$ through another rational function $\varphi := Y/X$ for $|\eta| \ge R$. For $\eta \ne 0$, $\varphi : L_\eta \to \mathbf{C}$ is written as $\varphi(\eta, y) = y/\eta$. Let $j_\theta : L_{\eta_0} \to L_{\eta_0 \exp(\theta i)}$, $0 \le \theta \le 2\pi$ be a family of homeomorphisms which is identity outside of a big disk under this identification $\varphi : L_\eta \to \mathbf{C}$. Then the base point b_0 stays constant under the identification by φ but under the first identification of $y : L_\eta \to \mathbf{P}^1$, the base point is rotated by $\theta \mapsto b_0 \exp(\theta i)$. Putting $h' = j_{2\pi}$, this implies that the monodromy relation around L_∞ is given by

$$h'_\sharp(g) = \omega g^{-\sigma_\infty} \omega^{-1}, \quad g \in G \tag{2.4.1}$$

This gives the following corollary.

Corollary (2.5). *Take another generic line $L_{\eta_0'}$ for C with $\eta_0' \neq \eta_0$. Let $R_1, \dots, R_\ell$ be the monodromy relation along σ_i as before. Then the fundamental group of a generic affine complement $\pi_1(\mathbf{P}^2 - C \cup L_{\eta_0'}; b_0)$ is isomorphic to the quotient group of $\pi_1(\mathbf{C}^2 - C^a; b_0)$ by the relation $\omega g_i = g_i \omega$, $i = 1, \dots, d$. In particular, if ω is in the center of $\pi_1(\mathbf{C}^2 - C^a; b_0)$, $\pi_1(\mathbf{C}^2 - C^a; b_0)$ is isomorphic to the fundamental group of a generic affine complement $\pi_1(\mathbf{P}^2 - C \cup L_{\eta_0'}; b_0)$.*

(C) Milnor fiber. Consider the affine hypersurface $V(C) = \{(x, y, z) \in \mathbf{C}^3; F(x, y, z) = 1\}$ where $F(X, Y, Z) = Z^d f(X/Z, Y/Z)$. The restriction of Hopf fibration to $V(C)$ is d-fold cyclic covering over $\mathbf{P}^2 - C$. Thus we have an exact sequence:

$$1 \to \pi_1(V(C)) \to \pi_1(\mathbf{P}^2 - C) \to \mathbf{Z}/d\mathbf{Z} \to 1 \tag{2.6}$$

Comparing with Hurewicz homomorphism, we get

Proposition (2.7) ([O2]). *If C is irreducible, $\pi_1(V(C))$ is isomorphic to the commutator group $\mathcal{D}(\pi_1(\mathbf{P}^2 - C))$ of $\pi_1(\mathbf{P}^2 - C)$.*

§3. Cyclic transforms of plane curves.

(A) Cyclic transforms. Let $C \subset \mathbf{P}^2$ be a projective curve of degree d. Fixing a line at infinity L_∞, we assume that the affine curve $C^a := C \cap \mathbf{C}^2$ is defined by $f(x, y) = 0$ in $\mathbf{C}^2 = \mathbf{P}^2 - L_\infty$. We assume that $f(x, y)$ is written with mutually distinct non-zero $\alpha_1, \dots, \alpha_k$ as

$$f(x, y) = \prod_{i=1}^{k} (y^a - \alpha_i x^b)^{\nu_i} + (\text{lower terms}), \qquad \gcd(a, b) = 1 \tag{$\sharp$}$$

Here (lower term) implies that it is a linear combination of monomials $x^\alpha y^\beta$ with $a\alpha + b\beta < kab$. This implies that $\deg_y f(x, y) = d'$, $\deg_x f(x, y) = d''$ where $d' := a \sum_{i=1}^k \nu_i$, $d'' := b \sum_{i=1}^k \nu_i$ and $d = \max(d', d'')$ and both pencils $\{x = \eta\}_{\eta \in \mathbf{C}}$ and $\{y = \delta\}_{\delta \in \mathbf{C}}$ are admissible. Note that the assumption ($\sharp$) does not change by the change of coordinates of the type $(x, y) \mapsto (x + \alpha, y + \beta)$.
(1) If $a = b = 1$, then $d = d' = d''$ and $L_\infty \cap C = \{[1; \alpha_i; 0]; i = 1, \dots, k\}$. In particular, if $\nu_i = 1$ for each i, L_∞ is generic for C and thus L_∞ intersects transversely with C.
(2) If $a > b$ (respectively $a < b$), we have $d = d'$, $C \cap L_\infty = \{\rho_\infty := [1; 0; 0]\}$ (resp. $d = d''$, $C \cap L_\infty = \{\rho'_\infty := [0; 1; 0]\}$) and C has a singularity at ρ_∞ (resp. at ρ'_∞). The local equation of C at ρ_∞ (resp. ρ'_∞) takes the form:

$$\begin{cases} \prod_{i=1}^k (\zeta^a - \alpha_i \xi^{a-b})^{\nu_i} + (\text{higher terms}) = 0, & \zeta = Y/X, \xi = Z/X, \ a > b \\ \prod_{i=1}^k ({\zeta'}^{b-a} - \alpha_i {\xi'}^b)^{\nu_i} + (\text{higher terms}) = 0, & \zeta' = Z/Y, \xi' = X/Y, \ a < b \end{cases} \tag{3.1}$$

Here (higher terms) is defined similarly. For instance, in the first equality it is a linear combinations of monomilas $\zeta^\alpha \xi^\beta$ with $(a - b)\alpha + a\beta > ka(a - b)$. Now we consider the horizontal pencil $M_\eta = \{y = \eta\}$, $\eta \in \mathbf{C}$ and let $D = M_\beta$ be a generic pencil line. As β is generic, $D \cap C^a$ is d'' distinct points in $\mathbf{C}^2$. For an integer $n \geq 2$, we consider the n-fold cyclic covering $\varphi_n : \mathbf{C}^2 \to \mathbf{C}^2$, defined by

$$\varphi_n : \mathbf{C}^2 \to \mathbf{C}^2, \quad \varphi_n(x, y) = (x, (y - \beta)^n + \beta)$$

which is branched along D. Let $\mathcal{C}_n(C;D)^a = \varphi_n^{-1}(C^a)$ and let $\mathcal{C}_n(C;D)$ be the closure of $\mathcal{C}_n(C;D)^a$ in $\mathbf{P}^2$. We call $\mathcal{C}_n(C;D)$ the *cyclic transform of order n with respect to the line D*. To avoid the confusion, we denote the source space of φ_n by $\widetilde{\mathbf{C}^2}$ and the coordinates of $\widetilde{\mathbf{C}^2}$ by $(\tilde{x},\tilde{y})$. Thus the line $\{\tilde{y}=\beta\}$ is equal to $\varphi_n^{-1}(D)$ and we denote it by $\widetilde{D}$. We denote the line at infinity $\mathbf{P}^2-\widetilde{\mathbf{C}^2}$ by $\widetilde{L}_\infty$. Let $f^{(n)}(\tilde{x},\tilde{y})$ be the defining polynomial of $\mathcal{C}_n(C;D)^a$. As $f^{(n)}(\tilde{x},\tilde{y}) = f(\tilde{x},(\tilde{y}-\beta)^n+\beta)$, $f^{(n)}(\tilde{x},\tilde{y})$ takes the form:

$$f^{(n)}(x,y) = \prod_{i=1}^{k}(\tilde{y}^{na}-\alpha_i\tilde{x}^b)^{\nu_i} + (\text{lower terms}). \tag{3.2}$$

Observer that $f^{(n)}(\tilde{x},\tilde{y})$ also satisfies ($\sharp$).

(B) Singularities of $\mathcal{C}_n(\mathrm{C};\mathrm{D})$. Let $\mathbf{a}_1,\dots,\mathbf{a}_s$ be the singular points of C^a and put $L_\infty\cap C = \{\mathbf{a}_\infty^1,\dots,\mathbf{a}_\infty^\ell\}$ and $\mathcal{C}_n(C;D)\cap\widetilde{L}_\infty = \{\tilde{\mathbf{a}}_\infty^i; i=1,\dots,\tilde{\ell}\}$ where $\widetilde{L}_\infty$ is the line at infinity of the projective compactification of the source space $\widetilde{\mathbf{C}^2}$ of φ_n. Note that $\ell=k$ if $a=b=1$ and $\ell=1$ otherwise. Note also that $\tilde{\ell}=kb$ or 1 according to $na=b$ or $na\neq b$. $\mathcal{C}_n(C;D)\cap\widetilde{L}_\infty$ is either $\{[1;0;0]\}$ if $na>b$ or $\{[0;1;0]\}$ if $na<b$. It is obvious that for each $i=1,\dots,s$, $\mathcal{C}_n(C;D)$ has n-copies of singularities $\mathbf{a}_{i,1},\dots,\mathbf{a}_{i,n}$ which are locally isomorphic to $\mathbf{a}_i$. We denote the local Milnor number at $\mathbf{a}\in C$ by $\mu(C;\mathbf{a})$. First we recall the modified Plücker's formula for the topological Euler characteristics (see, for instance,[O2]):

$$\chi(C) = 3d-d^2+\sum_{j=1}^{s}\mu(C;\mathbf{a}_j)+\sum_{i=1}^{\ell}\mu(C;\mathbf{a}_\infty^i) \tag{3.3.1}$$

Proposition (3.3.2). *If the branching locus D is a generic pencil line, the topological types of $(\widetilde{\mathbf{C}^2},\mathcal{C}_n(C;D)^a)$ and $(\mathbf{P}^2,\mathcal{C}_n(C;D))$ do not depend on the choice of a generic β.*

Note that $\mathcal{C}_n(C;D)$ has further singularities, if the branching line D is not generic.

(C) Main results of this section. Let G be an arbitrary group. We denote the commutator subgroup and the center of G by $\mathcal{D}(G)$ and $\mathcal{Z}(G)$ respectively. The main result of this section is :

Theorem (3.4). *Assume that ($\sharp$) is satisfied and D is a generic horizontal pencil line.*
(1) *The canonical homomorphism $\varphi_{n\sharp}:\pi_1(\widetilde{\mathbf{C}^2}-\mathcal{C}_n(C;D)^a)\to\pi_1(\mathbf{C}^2-C^a)$ is an isomorphism.*
(2-a) *Assume $a\geq b$ (so $\deg\mathcal{C}_n(C;D)=nd$). Then there is a surjective homomorphism $\Phi_n:\pi_1(\mathbf{P}^2-\mathcal{C}_n(C;D))\to\pi_1(\mathbf{P}^2-C)$ which gives the following commutative diagram.*

$$\begin{array}{ccc}
\pi_1(\mathbf{P}^2-\mathcal{C}_n(C;D)) & \xrightarrow{\Phi_n} & \pi_1(\mathbf{P}^2-C) \\
\uparrow{\scriptstyle\tilde{\iota}_\sharp} & & \uparrow{\scriptstyle\iota_\sharp} \\
\pi_1(\widetilde{\mathbf{C}^2}-\mathcal{C}_n(C;D)^a) & \xrightarrow{\varphi_{n\sharp}} & \pi_1(\mathbf{C}^2-C^a)
\end{array}$$

where $\tilde{\iota}_\sharp$ and $\iota_\sharp$ are induced by the respective inclusions and the kernel of Φ_n is normally generated by the class of $\omega' := \varphi_{n\sharp}^{-1}(\omega)$ where ω^{-1} is a lasso for L_∞ and ω'^{-n} is a lasso for the line at infinity $\widetilde{L}_\infty$ of $\widetilde{\mathbf{C}^2}$.
(2-b) *Assume that $na\leq b$ (so $\deg\mathcal{C}_n(C;D)=\deg C^a=d$). Then $\tilde{\omega}:=\varphi_{n\sharp}^{-1}(\omega)$ is a lasso for $\widetilde{L}_\infty$ and we have an isomorphism: $\pi_1(\mathbf{P}^2-\mathcal{C}_n(C;D))\cong\pi_1(\mathbf{P}^2-C)$.*

Corollary (3.4.1). *Assume that $a \geq b$ and L_∞ is central for C. Then*
(1) *$\widetilde{L}_\infty$ is central for $\mathcal{C}_n(C;D)$ and there is a canonical central extension of groups*

$$1 \to \mathbf{Z}/n\mathbf{Z} \xrightarrow{\iota} \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C;D)) \xrightarrow{\Phi_n} \pi_1(\mathbf{P}^2 - C) \to 1$$

(i.e., $\iota(\mathbf{Z}/n\mathbf{Z}) \subset \mathcal{Z}(\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C;D))))$ and $\mathbf{Z}/n\mathbf{Z}$ is generated by $\omega' = \varphi_{n\sharp}^{-1}(\omega)$.
(2) *The restriction of Φ_n gives an isomorphism of commutator groups*

$$\Phi_n : \mathcal{D}(\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C;D))) \to \mathcal{D}(\pi_1(\mathbf{P}^2 - C))$$

and the following exact sequences of the centers and the first homology groups:

$$\begin{array}{ccccccccc} 1 & \to & \mathbf{Z}/n\mathbf{Z} & \to & \mathcal{Z}(\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C;D))) & \xrightarrow{\Phi_n} & \mathcal{Z}(\pi_1(\mathbf{P}^2 - C)) & \to & 1 \\ 1 & \to & \mathbf{Z}/n\mathbf{Z} & \to & H_1(\mathbf{P}^2 - \mathcal{C}_n(C;D)) & \xrightarrow{\overline{\Phi}_n} & H_1(\mathbf{P}^2 - C) & \to & 1 \end{array}$$

Proof of Corollary (3.4.1). Assume that L_∞ is central. Then $\omega \in \mathcal{Z}(\pi_1(\mathbf{C}^2 - C^a; b_0))$. As $\varphi_{n\sharp}$ is an isomorphism, $\omega' \in \mathcal{Z}(\pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C); b_0^{(0)}))$. Thus the normal subgroup $\mathcal{N}(\omega')$ of $\pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C); b_0^{(0)})$ is simply the cyclic group $\langle\omega'\rangle$ generated by ω'. We consider the Hurewicz image of ω' in $H_1(\mathbf{P}^2 - \mathcal{C}_n(C))$. Suppose that C has r irreducible components C_j of degree d_j, $j = 1, \dots, r$. Then it is obvious that $\mathcal{C}_n(C)$ consists of r irreducible components $\mathcal{C}_n(C_1), \dots, \mathcal{C}_n(C_r)$ of degree $nd_1, \dots, nd_r$ respectively. For any fixed j, d_j-elements of $\{g_{1,j}, \dots, g_{d,j}\}$ are lassos for $\mathcal{C}_n(C_j)$. Thus ω' corresponds to the class $[\omega'] = (d_1, \dots, d_r)$ of $H_1(\mathbf{P}^2 - \mathcal{C}_n(C)) \cong \mathbf{Z}^r/(nd_1, \dots, nd_r)$. Thus $[\omega']$ has order n in the first homology group. As $\omega'^n = e$ already in $\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))$, order$(\omega') = n$ and the kernel of Φ_n is a cyclic group of order n generated by ω'. This proves the first assertion (1).

As Φ_n is surjective, the commutator subgroup $\mathcal{D}(\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C;D)))$ by Φ_n is mapped surjectively onto the commutator subgroup $\mathcal{D}(\pi_1(\mathbf{P}^2 - C))$. On the other hand, the kernel $\mathbf{Z}/n\mathbf{Z}$ is injectively mapped to the first homology group $H_1(\mathbf{P}^2 - \mathcal{C}_n(C))$. Thus $\mathcal{D}(\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))) \cap \mathbf{Z}/n\mathbf{Z} = \{e\}$. Therefore Φ_n induces an isomorphism of the commutator groups. The sequence

$$1 \to \mathbf{Z}/n\mathbf{Z} \to \mathcal{Z}(\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))) \xrightarrow{\Psi'_n} \mathcal{Z}(\pi_1(\mathbf{P}^2 - C))$$

is clearly exact. We show the surjectivity of Ψ'_n. Take $h' \in \mathcal{Z}(\pi_1(\mathbf{P}^2 - C))$ and choose $h \in \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))$ so that $\Phi_n(h) = h'$. For any $g \in \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))$, the image of the commutator $hgh^{-1}g^{-1}$ by Φ_n is trivial. Thus we can write $hgh^{-1}g^{-1} = \omega'^a$ for some $0 \leq a \leq n-1$. As $[\omega']$ has order n in first homology, this implies that $a = 0$ and thus $hg = gh$ for any g. Therefore h is in the center. The last exact sequence of the assertion (2) follows by a similar argument. This completes the proof of Corollary (3.4.1). □

Remark (3.5). (1) We remark that the rational map $\varphi'_n : \mathbf{P}^2 \to \mathbf{P}^2$ which is associated with φ_n is defined by $\varphi'_n([X;Y;Z]) = [XZ^{n-1}; Y^n; Z^n]$ and thus φ'_n is not defined at $\rho_\infty := [1;0;0] \in \mathcal{C}_n(C)$ and $\varphi'_n(\widetilde{L}_\infty - \{\rho_\infty\}) = \rho'_\infty = [0;1;0]$.
(2) In the case of $na > b > a$, there does not exist a surjective homomorphism $\Phi_n : \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C)) \to \pi_1(\mathbf{P}^2 - C)$ in general. For example, take C' a smooth curve of degree d' and let $C = \mathcal{C}_2(C';D')$ a generic two fold covering with respect to a generic line $D' := \{x = \alpha\}$. Then we take a covering $\mathcal{C}_3(C;D)$ of degree 3 with respect to a generic $D := \{y = \beta\}$. Then we know that $\deg C = 2d'$ and $\deg \mathcal{C}_3(C;D) = 3d'$ and therefore $\pi_1(\mathbf{P}^2 - \mathcal{C}_3(C;D)) = \mathbf{Z}/3d'\mathbf{Z}$ and $\pi_1(\mathbf{P}^2 - \mathcal{C}_2(C';D')) = \mathbf{Z}/2d'\mathbf{Z}$. Thus there does not exist any surjective homomorphism.

(D) Generic cyclic covering. Now we consider the generic case:

$$f(x,y) = \prod_{i=1}^{d}(y-\alpha_i x) + (\text{lower terms}), \quad \alpha_1,\dots,\alpha_d \in \mathbf{C}^*, \quad \alpha_i \neq \alpha_j \ (i \neq j) \tag{3.6}$$

This is always the case if we choose the line at infinity L_∞ to be generic and then generic affine coordinates (x,y). Take positive integers $n \geq m \geq 1$ and we denote $\mathcal{C}_n(C;D)$ by $\mathcal{C}_n(C)$ and $\mathcal{C}_m(\mathcal{C}_n(C;D);D')$ by $\mathcal{C}_{m,n}(C)$ where $D=\{y=\beta\}$ and $D'=\{x=\alpha\}$ with generic α,β. Note that $\mathcal{C}_n(C)=\mathcal{C}_{1,n}(C)$. The topology of the complement of $\mathcal{C}_{m,n}(C)$ depends only on C and m,n. We will refer $\mathcal{C}_n(C)$ and $\mathcal{C}_{m,n}(C)$ as *a generic n-fold* (respectively *a generic (m,n)-fold*) *covering transform* of C. They are defined in $\mathbf{C}^2$ by

$$\mathcal{C}_n(C)^a = \{(\tilde{x},\tilde{y}) \in \mathbf{C}^2; f(\tilde{x},\tilde{y}^n)=0\}, \quad \mathcal{C}_{m,n}(C)^a = \{(\tilde{x},\tilde{y}) \in \mathbf{C}^2; f(\tilde{x}^m,\tilde{y}^n)=0\}$$

taking a change of coordinate $(x,y)\mapsto(x+\alpha,y+\beta)$ if necessary. If $n>m$, $\mathcal{C}_{m,n}(C)$ has only one singularity at $\rho_\infty=[1;0;0]$ and the local equation takes the following form:

$$\prod_{i=1}^{d}(\zeta^n-\alpha_i\xi^{n-m}) + (\text{higher terms}) = 0, \quad \zeta = Y/X, \xi = Z/X$$

Therefore $\mathcal{C}_{m,n}(C)$ is locally $d\times\gcd(m,n)$ irreducible components at ρ_∞. $(\mathcal{C}_{m,n}(C),\rho_\infty)$ is topologically equivalent to the germ of a Brieskorn singularity $B((n-m)d,nd)$ where $B(p,q):=\{\xi^p-\zeta^q\}=0$. In the case $m=n$, we have no singularity at infinity. By Theorem (3.4) and Corollary (3.4.1), we have the following.

Theorem (3.7). *Let $\mathcal{C}_n(C)$ and $\mathcal{C}_{m,n}(C)$ be as above. Then the canonical homomorphisms*

$$\pi_1(\widetilde{\widetilde{\mathbf{C}^2}} - \mathcal{C}_{m,n}(C)^a) \xrightarrow{\varphi_{m\sharp}} \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C)^a) \xrightarrow{\varphi_{n\sharp}} \pi_1(\mathbf{C}^2 - C^a)$$

and $\Phi_m : \pi_1(\mathbf{P}^2 - \mathcal{C}_{m,n}(C)) \to \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))$ are isomorphisms. There exist canonical central extensions of groups where the diagrams are commutative.

$$\begin{array}{ccccccccc}
1 & \to & \mathbf{Z}/n\mathbf{Z} & \xrightarrow{\iota} & \pi_1(\mathbf{P}^2-\mathcal{C}_{m,n}(C)) & \xrightarrow{\Phi_{m,n}} & \pi_1(\mathbf{P}^2-C) & \to & 1 \\
 & & \downarrow \mathrm{id} & \circlearrowleft & \cong \downarrow \Phi_m & \circlearrowleft & \downarrow \mathrm{id} & & \\
1 & \to & \mathbf{Z}/n\mathbf{Z} & \xrightarrow{\iota'} & \pi_1(\mathbf{P}^2-\mathcal{C}_n(C)) & \xrightarrow{\Phi_n} & \pi_1(\mathbf{P}^2-C) & \to & 1
\end{array}$$

The kernel $\operatorname{Ker}\Phi_n$ (respectively $\operatorname{Ker}\Phi_{m,n}$) is generated by an element ω' (resp. $\omega''=\Phi_m^{-1}(\omega')$) in the center such that ω'^n (resp. ω''^n) is a lasso for $\widetilde{L}_\infty$ (resp. for $\widetilde{\widetilde{L}}_\infty$). The restriction of $\Phi_{m,n}$, Φ_m and Φ_n give an isomorphism of the respective commutator groups

$$\Phi_{m,n,\mathcal{D}} : \mathcal{D}(\pi_1(\mathbf{P}^2-\mathcal{C}_{m,n}(C))) \xrightarrow{\Phi_{m,\mathcal{D}}} \mathcal{D}(\pi_1(\mathbf{P}^2-\mathcal{C}_n(C))) \xrightarrow{\Phi_{n,\mathcal{D}}} \mathcal{D}(\pi_1(\mathbf{P}^2-C))$$

and exact sequences of the centers and the first homology groups:

$$\begin{array}{ccccccccc}
1 & \to & \mathbf{Z}/n\mathbf{Z} & \to & \mathcal{Z}(\pi_1(\mathbf{P}^2-\mathcal{C}_{m,n}(C))) & \xrightarrow{\Phi_{m,n}} & \mathcal{Z}(\pi_1(\mathbf{P}^2-C)) & \to & 1 \\
1 & \to & \mathbf{Z}/n\mathbf{Z} & \to & H_1(\mathbf{P}^2-\mathcal{C}_{m,n}(C)) & \xrightarrow{\overline{\Phi}_{m,n}} & H_1(\mathbf{P}^2-C) & \to & 1
\end{array}$$

Let $\{\mathbf{a}_1,\dots,\mathbf{a}_s\}$ be singular points as before. Then $\mathcal{C}_n(C)$ (respectively $\mathcal{C}_{m,n}(C)$) has n copies (resp. nm copies) of $\mathbf{a}_i$ for each $i=1,\dots,s$ and one singularity at $\rho_\infty:=[1;0;0]$ except the case $n=m$. The curve $\mathcal{C}_{n,n}(C)$ has no singularity at infinity. The similar assertion for $\mathcal{C}_{n,n}(C)$ is obtained independently by Shimada [Sh].

Corollary (3.7.1). (1) *$\pi_1(\mathbf{P}^2 - \mathcal{C}_{m,n}(C))$ is abelian if and only if $\pi_1(\mathbf{P}^2 - C)$ is abelian.*
(2) *Assume that C is irreducible. Then the fundamental groups $\pi_1(V(\mathcal{C}_{m,n}(C)))$ and $\pi_1(V(C))$ of the respective Milnor fibers $V(\mathcal{C}_{m,n}(C))$ of $\mathcal{C}_{m,n}(C)$ and $V(C)$ of C are isomorphic.*

Proof. The assertion (1) follows from Theorem (3.7). The assertion (2) is immediate from Proposition (2.7) and Theorem (3.7). □

The following is also an immediate consequence of Theorem (3.7) and Corollary (2.5).

Corollary (3.7.2). *$\tilde{\tilde{L}}_\infty$ is central for $\mathcal{C}_{m,n}(C)$ i.e., $\pi_1(\mathbf{P}^2 - \mathcal{C}_{m,n}(C) \cup \tilde{\tilde{L}}_\infty)$ is isomorphic to the fundamental group of the generic affine complement of $\mathcal{C}_{m,n}(C)$.*

(E) Homologically injectivity condition of the center. The following is useful to produce Zariski pairs from a given Zariski pair (See §5). First we consider the following condition for a group G:

$$\text{(H.I.C)} \qquad \mathcal{Z}(G) \cap \mathcal{D}(G) = \{e\}$$

This is equivalent to the injectivity of the composition: $\mathcal{Z}(G) \hookrightarrow G \to H_1(G) := G/\mathcal{D}(G)$. When this condition is satisfied, we say that G satisfies *homological injectivity condition of the center* (or (H.I.C)-condition in short).

Theorem (3.8). *Let $C = C_1 \cup \cdots \cup C_r$ and $C' = C_1' \cup \cdots \cup C_r'$ be projective curves with the same number of irreducible components and assume that $\mathrm{degree}(C_i) = \mathrm{degree}(C_i') = d_i$ for $i = 1, \dots, r$ and assume that $\pi_1(\mathbf{P}^2 - C')$ satisfies (H.I.C)-condition. Assume that $\pi_1(\mathbf{P}^2 - \mathcal{C}_{m,n}(C))$ and $\pi_1(\mathbf{P}^2 - \mathcal{C}_{m,n}(C'))$ are isomorphic for some integer m, n with $1 \le m \le n$. Then $\pi_1(\mathbf{P}^2 - C)$ and $\pi_1(\mathbf{P}^2 - C')$ are also isomorphic.*

Remark (3.9). (1) Take a non-generic line $D = \{y = \beta\}$ for C and consider the corresponding cyclic covering branched along D, $\varphi_n : \mathbf{C}^2 \to \mathbf{C}^2$. Then the assertions in Theorem (3.4) and Corollary (3.4.1) for the pull back $C' = \varphi_n^{-1}(C)$ may fail in general. For example, we can take the quartic defined by (5.1.1) in §5. Then L_∞ is central for C and $\pi_1(\mathbf{P}^2 - C) = \mathbf{Z}/4\mathbf{Z}$. Take $D = \{y = 0\}$ and consider $\varphi_2 : \mathbf{C}^2 \to \mathbf{C}^2$, $\varphi_2(x, y) = (x, y^2)$. Then the pull back Z_4 of C is a so called Zariski's three cuspidal quartic and $\pi_1(\mathbf{P}^2 - Z_4)$ ia a finite non-abelian group of order 12 ([Z1],[O5]). See also §5.
(2) We do not have any example of a plane curve C such that $\pi_1(\mathbf{P}^2 - C)$ does not satisfy the (H.I.C)-condition.

§4. Jung transforms of plane curves.

Let C be a projective curve of degree d in $\mathbf{P}^2$ and let $f(x, y) = 0$ be the defining polynomial of C with respect to the affine space $\mathbf{C}^2 = \mathbf{P}^2 - L_\infty$. In this section, we introduce another operation which produces a projective curve $\mathcal{J}_n(C)$ of degree nd.

(A) Jung transform of degree n. First for any integer $n \ge 2$ we consider the following automorphism of $\mathbf{C}^2$ ([J]).

$$(4.1) \qquad J_n : \mathbf{C}^2 \to \mathbf{C}^2, \quad J_n(x, y) = (x + y^n, y).$$

The inverse of J_n is given by $J_n^{-1}(x,y) = (x - y^n, y)$. Let $\mathcal{J}_n(C; L_\infty)$ be the projective closure of $J_n^{-1}(C^a)$. We call $\mathcal{J}_n(C; L_\infty)$ an *Jung transform* of C of degree n. By the definition, $\mathcal{J}_n(C; L_\infty)$ is birationally equivalent to C and the affine complements $\mathbf{C}^2 - C^a$ and $\mathbf{C}^2 - \mathcal{J}_n(C; L_\infty)^a$ are biholomorphic. We denote the source space of J_n by $\widetilde{\mathbf{C}^2}$, the line at infinity by $\widetilde{L}_\infty$ and the affine coordinates by $(\tilde{x}, \tilde{y})$ as in §3. By the definition, $\mathcal{J}_n(C; L_\infty)$ is defined in $\widetilde{\mathbf{C}^2}$ by

$$f^{(n)}(\tilde{x}, \tilde{y}) = f(\tilde{x} + \tilde{y}^n, \tilde{y}). \tag{4.2}$$

We say that J_n or the affine coordinates (x, y) is *an admissible* for C if $[1;0;0] \notin C$. We call $\mathcal{J}_n(C; L_\infty)$ an *admissible Jung transform* of C of degree n if J_n is admissible. Note that the admissibility of $\mathcal{J}_n$ implies that $\deg f^{(n)}(\tilde{x}, \tilde{y}) = nd$. Finally we call $\mathcal{J}_n(C; L_\infty)$ a *generic* Jung transform of C of degree n, if L_∞ is generic with respect to C and J_n is admissible for C. In this case, we denote $\mathcal{J}_n(C; L_\infty)$ simply by $\mathcal{J}_n(C)$.

(B) Singularities of $\mathcal{J}_n(\mathbf{C}; \mathbf{L}_\infty)$. We consider the singularities of an admissible Jung transform $\mathcal{J}_n(C; L_\infty)$. Let $\mathbf{a}_1, \dots, \mathbf{a}_s$ be the singular points of C^a and let $\{\mathbf{a}_\infty^1, \dots, \mathbf{a}_\infty^k\} = C \cap L_\infty$ be the points at infinity. Let r_i be the number of local irreducible components of C at $\mathbf{a}_\infty^i$. As J_n is biholomorphic, the singularities of $\mathcal{J}_n(C; L_\infty)$ in $\mathbf{C}^2$ corresponds bijectively to $\mathbf{a}_1, \dots, \mathbf{a}_s$. Let $f(x,y) = f_d(x,y) + f_{d-1}(x,y) + \cdots + f_0$ be the homogeneous decomposition of f. By admissibility, we can write $f_d(x,y) = \prod_{i=1}^k (x - \alpha_i y)^{\nu_i}$ where $\alpha_1, \dots, \alpha_d \in \mathbf{C}$ are mutually distinct and $\sum_{i=1}^k \nu_i = d$. We may assume that $\mathbf{a}_\infty^i = (\alpha_i; 1; 0)$ in the homogeneous coordinates. Then the homogeneous polynomial which defines $\mathcal{J}_n(C; L_\infty)$ is

$$F^{(n)}(X,Y,Z) := \prod_{i=1}^k (XZ^{n-1} + Y^n - \alpha_i Y Z^{n-1})^{\nu_i} + \sum_{j=1}^d Z^{jn} f_{d-j}(XZ^{n-1} + Y^n, YZ^{n-1}) \tag{4.3}$$

Thus $\deg \mathcal{J}_n(C; L_\infty) = nd$ and $\rho_\infty := [1;0;0]$ is the only intersection of $\mathcal{J}_n(C; L_\infty)$ with the line at infinity $\widetilde{L}_\infty$ and ρ_∞ is a singular point of $\mathcal{J}_n(C; L_\infty)$. The number of local irreducible components of $\mathcal{J}_n(C; L_\infty)$ at ρ_∞ is $\sum_{i=1}^k r_i$ and the local Milnor number $\mu(\mathcal{J}_n(C; L_\infty); \mathbf{a}_\infty)$ can be computed using the modified Plücker's formula :

$$\chi(\mathcal{J}_n(C; L_\infty)) = 3nd - n^2 d^2 + \sum_{i=1}^s \mu(C; \mathbf{a}_i) + \mu(\mathcal{J}_n(C; L_\infty); \mathbf{a}_\infty) = \chi(C) - k + 1 \tag{4.4}$$

Thus the Milnor number $\mu(\mathcal{J}_n(C; L_\infty); \mathbf{a}_\infty)$ is independent of the choice of the admissible affine coordinate (x, y) of $\mathbf{C}^2 = \mathbf{P}^2 - L_\infty$. As the space of the admissible affine coordinates are connected and a μ-constant family of plane curves are topologically equivalent to each other, we have:

Proposition (4.5). *The topological type of the pair $(\mathbf{P}^2, \mathcal{J}_n(C; L_\infty))$ depend only on C and L_∞ and it does not depend on the choice of the admissible affine coordinates (x, y). If L_∞ is generic, the topological type of the pair $(\mathbf{P}^2, \mathcal{J}_n(C; L_\infty))$ does not depend on L_∞.*

Let us study the structure of the singularity $\rho_\infty \in \mathcal{J}_n(C)$ of a generic admissible Jung transform of degree n in detail. Let $\zeta = Y/X, \xi = Z/X$ be affine coordinates centered at ρ_∞ of the affine space $\mathbf{P}^2 - \{X = 0\}$. Then local defining polynomial takes the following form:

$$h(\zeta, \xi) = \prod_{i=1}^d (\xi^{n-1} + \zeta^n - \alpha_i \zeta \xi^{n-1}) + \sum_{j=1}^d \xi^{jn} f_{d-j}(\xi^{n-1} + \zeta^n, \zeta \xi^{n-1}) \tag{4.6}$$

$\mathcal{J}_n(C)$ has d irreducible components at ρ_∞. Consider an admissible toric modification

$$\pi : \mathbf{C}^2 \to \mathbf{C}^2, \quad \pi(u,v) = (\zeta,\xi), \quad \zeta = uv^{n-1},\ \xi = uv^n.$$

Then the defining polynomial changes into

$$\pi^* h(u,v) = v_1^{dn(n-1)}(-1)^{d(n-1)}\left(\prod_{i=1}^{d}(u_1 + \alpha_i v_1^{n-1}) + (\text{higher terms})\right)$$

where $u_1 := u+1, v_1 := v$ are local coordinates at $(u,v) = (-1,0)$. Thus we see that the Newton boundary of $\pi^* h$ in (u_1, v_1) is non-degenerate. Thus the resolution complexity $\varrho(\mathcal{J}_n(C);\rho_\infty)$ is two for $n \geq 3$. See [Le-Oka] for the definition of the resolution complexity. The Milnor number is given by $\mu(\mathcal{J}_n(C);\rho_\infty) = d^2(n^2-1) - d(3n-2) + 1$. (In the case of $n = 2$, the resolution complexity $\varrho(\mathcal{J}_n(C);\rho_\infty)$ is 1.) The germ $(\mathcal{J}_n(C);\rho_\infty)$ is topologically determened by the first term of (4.6) and it is equivalent to $B(n-1,n;d) := \{(\xi^{n-1}+\zeta^n)^d - (\zeta\xi^{n-1})^d = 0\}$.

(C) Main results of this section. Now we state the main result of this section.

Theorem (4.7). *Assume that L_∞ is central for C and let $J_n : \widetilde{\mathbf{C}^2} \to \mathbf{C}^2$ be an admissible Jung transform of degree n of C. Then $\widetilde{L}_\infty$ is central for $\mathcal{J}_n(C;L_\infty)$ and there exists a unique surjective homomorphism $\Psi_n : \pi_1(\mathbf{P}^2 - \mathcal{J}_n(C;L_\infty)) \to \pi_1(\mathbf{P}^2 - C)$ which gives the following commutative diagram*

$$\begin{array}{ccc}
\pi_1(\mathbf{P}^2 - \mathcal{J}_n(C;L_\infty)) & \xrightarrow{\Psi_n} & \pi_1(\mathbf{P}^2 - C) \\
\uparrow{\scriptstyle \widetilde{\iota}_\sharp} & & \uparrow{\scriptstyle \iota_\sharp} \\
\pi_1(\widetilde{\mathbf{C}^2} - \mathcal{J}_n(C;L_\infty)^a) & \xrightarrow{J_{n\sharp}} & \pi_1(\mathbf{C}^2 - C^a)
\end{array}$$

where $\widetilde{\iota}_\sharp$ and $\iota_\sharp$ are associated with the respective inclusion maps. Ψ_n has the following property.
(1) The kernel of Ψ_n is a cyclic group of order n which is a subgroup of the center. So we have a central exactension of groups:

$$1 \to \mathbf{Z}/n\mathbf{Z} \xrightarrow{\alpha} \pi_1(\mathbf{P}^2 - \mathcal{J}_n(C;L_\infty)) \xrightarrow{\Psi_n} \pi_1(\mathbf{P}^2 - C) \to 1$$

The image $\alpha(\mathbf{Z}/n\mathbf{Z})$ is generated by $\widetilde{\iota}_\sharp(\omega')$ where $\omega' := J_{n\sharp}^{-1}(\omega)$, ω is a lasso for L_∞ in the base space $\mathbf{P}^2 \supset C$, and ω'^n is a lasso for the line at infinity $\widetilde{L}_\infty$.
(2) The restriction of Ψ_n gives an isomorphism $\Psi_n : \mathcal{D}(\pi_1(\mathbf{P}^2 - \mathcal{J}_n(C;L_\infty))) \to \mathcal{D}(\pi_1(\mathbf{P}^2 - C))$ and the following exact sequences of the centers and the first homology groups:

$$\begin{array}{ccccccccc}
1 & \to & \mathbf{Z}/n\mathbf{Z} & \to & \mathcal{Z}(\pi_1(\mathbf{P}^2 - \mathcal{J}_n(C;L_\infty))) & \xrightarrow{\Psi_n} & \mathcal{Z}(\pi_1(\mathbf{P}^2 - C)) & \to & 1 \\
1 & \to & \mathbf{Z}/n\mathbf{Z} & \to & H_1(\mathbf{P}^2 - \mathcal{J}_n(C;L_\infty)) & \xrightarrow{\overline{\Psi}_n} & H_1(\mathbf{P}^2 - C) & \to & 1
\end{array}$$

The proof is parallel to that of Theorem (3.4). See [O7]. The essencial poit is:

Lemma (4.7.4). *$J_{n\sharp}(\widetilde{\omega}) = \omega^n$, $\omega'^n = \widetilde{\omega}$ and the order of $\widetilde{\iota}_\sharp(\omega')$ in $\pi_1(\mathbf{P}^2 - \mathcal{J}_n(C;L_\infty))$ is n.*

Assuming this for a moment, we complete the proof of Theorem (4.7). As $J_{n\sharp}$ is an isomorphism, $\omega' \in \mathcal{Z}(\pi_1(\widetilde{\mathbf{C}^2} - \mathcal{J}_n(C;L_\infty);\widetilde{b}_0))$ and $\pi_1(\mathbf{P}^2 - \mathcal{J}_n(C;L_\infty);\widetilde{b}_0) \cong \pi_1(\mathbf{C}^2 - C^a;b_0)/\langle\omega^n\rangle$ by (4.7.3). Combining this with (4.7.2), we get a central extension

$$1 \to \langle\widetilde{\iota}_\sharp(\omega')\rangle \to \pi_1(\mathbf{P}^2 - \mathcal{J}_n(C;L_\infty);\widetilde{b}_0) \xrightarrow{\Psi_n} \pi_1(\mathbf{P}^2 - C;b_0) \to 1$$

where Ψ_n is the quotient homomorphism which is associated with the above identification. This proves (1). The assertion (2) can be proved by the exact same way as in the proof of Corollary (3.4.1).

(D) Corollaries. The proofs of the following Corollaries are given by the exact same way as those of Corollaries (3.7.1), (3.7.2) and Theorem (3.8).

Corollary (4.8). *Let $J_n : \mathbf{C}^2 \to \mathbf{C}^2$ be an admissible Jung transform of degree n with respect to a central line at infinity L_∞. Then we have the following.*
(1) $\pi_1(\mathbf{P}^2 - \mathcal{J}_n(C; L_\infty))$ is abelian if and only if $\pi_1(\mathbf{P}^2 - C)$ is abelian.
(2) Assume that C is irreducible. Then $\pi_1(V(\mathcal{J}_n(C; L_\infty))) \cong \pi_1(V(C))$ where $V(\mathcal{J}_n(C; L_\infty))$ and $V(C)$ are respective Milnor fibers of $\mathcal{J}_n(C; L_\infty)$ and C.

Corollary (4.9). *Let $J_n : \mathbf{C}^2 \to \mathbf{C}^2$ be an admissible Jung transform of degree n with respect to a central line at infinity L_∞. Then $\widetilde{L}_\infty$ is central for $\mathcal{J}_n(C; L_\infty)$ and $\pi_1(\mathbf{P}^2 - \mathcal{J}_n(C; L_\infty) \cup \widetilde{L}_\infty)$ is isomorphic to the fundamental group of a generic affine complement of $\mathcal{J}_n(C; L_\infty)$.*

Corollary (4.10). *Let $J_n : \mathbf{C}^2 \to \mathbf{C}^2$ be an admissible Jung transform of degree n with respect to a central line at infinity L_∞. Let $C = C_1 \cup \cdots \cup C_r$ and $C' = C_1' \cup \cdots \cup C_r'$ be projective curves with the same number of irreducible components and assume that* degree(C_i) = degree(C_i') $= d_i$ *for $i = 1, \dots, r$. We assume that either $\pi_1(\mathbf{P}^2 - C)$ or $\pi_1(\mathbf{P}^2 - C')$ satisfies (H.I.C)-condition and that $\pi_1(\mathbf{P}^2 - \mathcal{J}_n(C; L_\infty))$ and $\pi_1(\mathbf{P}^2 - \mathcal{J}_n(C'))$ are isomorphic. Then $\pi_1(\mathbf{P}^2 - C)$ and $\pi_1(\mathbf{P}^2 - C')$ are isomorphic.*

Remark (4.11). (1) In the definition of an admissible Jung transform, we can take an affine automorphism

$$J_n' : \mathbf{C}^2 \to \mathbf{C}^2, \quad (x, y) \mapsto (x + h_n(y), y)$$

where $h_n(y)$ is an arbitrary polynomial of degree n. Let $\mathcal{J}_n'(C; L_\infty)$ be the closure of ${J_n'}^{-1}(C^a)$. Then the topological type of the pair $(\mathbf{P}^2, \mathcal{J}_n'(C; L_\infty))$ is equal to that of $(\mathbf{P}^2, \mathcal{J}_n(C; L_\infty))$.
(2) If $J_n : \mathbf{C}^2 \to \mathbf{C}^2$ is admissible but L_∞ is not necessarily central, there exists a surjective homomorphism $\Psi_n : \pi_1(\mathbf{P}^2 - \mathcal{J}_n(C; L_\infty)) \to \pi_1(\mathbf{P}^2 - C)$. In fact, assuming the admissibility $[1;0;0] \notin C$, J_n ca be extended a birational mapping $J_n' : \mathbf{P}^2 \to \mathbf{P}^2$ defined by $J_n'([X;Y;Z]) = [XZ^{n-1} + Y^n; YZ^{n-1}; Z^n]$. J_n' is well-defined on $\mathbf{P}^2 - \{[1;0;0]\}$ and $J_n'(\widetilde{L}_\infty - \{[1;0;0]\}) = [1;0;0]$. So $J_n' : \mathbf{P}^2 - \mathcal{J}_n(C; L_\infty)) \to \mathbf{P}^2 - C$ is well-defined. However $\operatorname{Ker}\Psi_n$ is not necessarily a cyclic group of order n.

§5. Zariski's quartic and Zariski pairs.

In this section, we apply the results of §3 and §4 to construct plane curves whose complement have interesting fundamental groups.

(A) Zariski's three cuspidal quartics. Let Z_4 be an irreducible quartic with three cusps. Such a curve is a rational curve. For example, we can take the following curve which is defined in $\mathbf{C}^2$ by the following equation ([O6]):

$$Z_4^a = \{(x, y) \in \mathbf{C}^2; x^3(3x + 8) - 6x^2(y^2 - 1) - (y^2 - 1)^2 = 0\} \tag{5.1}$$

We call such a curve *a Zariski's three cuspidal quartic.* It is known that the fundamental group $\pi_1(\mathbf{C}^2 - Z_4)$ and $\pi_1(\mathbf{P}^2 - Z_4)$ have the following representations ([Z1],[O6]):

$$\text{(5.2)} \qquad \begin{cases} \pi_1(\mathbf{C}^2 - Z_4) & = \langle \rho, \xi; \{\rho, \xi\} = e, \ \rho^2 = \xi^2 \rangle \\ \pi_1(\mathbf{P}^2 - Z_4) & = \langle \rho, \xi; \{\rho, \xi\} = e, \ \rho^2 = \xi^2, \rho^4 = e \rangle \end{cases}$$

where ρ and ξ are lassos for C and $\{\rho, \xi\} := \rho\xi\rho\xi^{-1}\rho^{-1}\xi^{-1}$. The relation $\{\rho, \xi\} = e$ is equivalent to $\rho\xi\rho = \xi\rho\xi$. A lasso ω for L_∞ is given by $\rho^2\xi^2(= \rho^4)$. Recall that ω^{-1} is a lasso for L_∞ and is contained in the center. A Zariski's three cuspidal quartic is the first example whose complement has a non-abelian finite fundamental group. We first recall the proof of the finiteness.

Lemma (5.3) ([Z1]). *Put*

$$G_1 = \langle \rho, \xi; \{\rho, \xi\} = e, \ \rho^2 = \xi^2, \rho^4 = e \rangle.$$

Then G_1 is a finite group of order 12 such that $\mathcal{D}(G_1) = \langle \rho^2\xi\rho \rangle \cong \mathbf{Z}/3\mathbf{Z}$, $\mathcal{Z}(G_1) = \langle \rho^2 \rangle \cong \mathbf{Z}/2\mathbf{Z}$ and $H_1(G_1) \cong \mathbf{Z}/4\mathbf{Z}$ and it is generated by the class of ρ

We consider the Hurewicz exact sequence:

$$\text{(5.4)} \qquad 1 \to \mathcal{D}(G_1) \cong \mathbf{Z}/3 \xrightarrow{\iota_1} G_1 \xrightarrow{\psi} H_1(G_1) \cong \mathbf{Z}/4\mathbf{Z} \to 1$$

This sequence splits by taking the section $\bar{\rho} \mapsto \rho$ of ψ so that G_1 has a structure of a semi-direct product of $\mathbf{Z}/3\mathbf{Z}$ and $\mathbf{Z}/4\mathbf{Z}$. More precisely, the semi-direct structure is given by $\rho\beta\rho^{-1} = \beta^2$ as $\rho\beta\rho^{-1} = \rho(\rho^2\xi\rho)\rho^{-1} = \rho^3\xi = \beta^2$.

(B) Generic transforms of a Zariski's quartic. Let $\mathcal{C}_n(Z_4)$ (respectively $\mathcal{C}_{n,n}(Z_4)$) be a generic cyclic transform of degree n (resp. of (n, n)) of the Zariski's quartic Z_4 and let $\mathcal{J}_n(Z_4)$ be a generic Jung transform of degree n of the Zariski's quartic Z_4. The singularities of $\mathcal{C}_n(Z_4)$ (respectively of $\mathcal{C}_{n,n}(Z_4)$) are $3n$ cusps (resp. $3n^2$ cusps). $\mathcal{C}_n(Z_4)$ has one more singularity at $\rho_\infty \in L_\infty$ and $(\mathcal{C}_n(Z_4), \rho_\infty)$ is equal to $B((n-1)d, nd) := \{\zeta^{nd} - \xi^{d(n-1)}\} = 0\}$. On the other hand, $\mathcal{J}_n(Z_4)$ is a rational curve which has 3 cusps and one more singularity at infinity $\rho_\infty \in \mathcal{J}_n(Z_4) \cap L_\infty$. $(\mathcal{J}_n(Z_4), \rho_\infty)$ is topologically equal to $B(n-1, n; d) := \{(\xi^{n-1} + \zeta^n)^d - (\zeta\xi^{n-1})^d = 0\}$. By Corollary (3.4.1) and Theorem (4.7), we have the following:

Theorem (5.5). *The affine fundamental groups $\pi_1(\mathbf{C}^2 - \mathcal{C}_n(Z_4)^a)$, $\pi_1(\mathbf{C}^2 - \mathcal{J}_n(Z_4)^a)$ are isomorphic to $\pi_1(\mathbf{C}^2 - Z_4) \cong \langle \rho_n, \xi_n; \{\rho_n, \xi_n\} = e, \ \rho_n^2 = \xi_n^2 \rangle$.*
(1) The projective fundamental groups $\pi_1(\mathbf{P}^2 - \mathcal{C}_n(Z_4))$ and $\pi_1(\mathbf{P}^2 - \mathcal{J}_n(Z_4))$ are isomorphic to G_n where G_n is defined by $G_n := \langle \rho_n, \xi_n; \{\rho_n, \xi_n\} = e, \ \rho_n^2 = \xi_n^2, \rho_n^{4n} = e \rangle$. Moreover we have a central extension of groups:

$$\text{(5.5.1)} \qquad 1 \to \mathbf{Z}/n\mathbf{Z} \to G_n \xrightarrow{\Phi_n} G_1 \to 1$$

defined by $\Phi_n(\rho_n) = \rho$ and $\Phi_n(\xi_n) = \xi$ and $\operatorname{Ker}\Phi_n$ is generated by ρ_n^4. In particular, we have $|G_n| = 12n$, $\mathcal{D}(G_n) = \langle \beta_n \rangle \cong \mathbf{Z}/3\mathbf{Z}$ where $\beta_n = [\rho_n, \xi_n]$ and $\mathcal{Z}(G_n) = \langle \rho_n^2 \rangle \cong \mathbf{Z}/2n\mathbf{Z}$.
(2) The Hurewicz sequence $1 \to \mathcal{D}(G_n) \to G_n \to H_1(G_n) \to 1$ has a canonical cross section $\theta : H_1(G_n) \to G_n$ which is given by $\theta(\bar{\rho}_n) = \rho_n$. This gives G_n a structure of semi-direct product $\mathbf{Z}/3$ and $\mathbf{Z}/4n\mathbf{Z}$ which is determined by $\rho_n\beta_n\rho_n^{-1} = \beta_n^2$.
(3) G_n is identified with the subgroup of the permutation group $\mathfrak{S}_{12n}$ of $12n$ elements $\{x_i, y_j, z_k; 1 \le$

$i, j, k \le 4n\}$ *generated by two permutations:* $\sigma_n = (x_1, \dots, x_{4n})(y_1, \dots, y_{4n})(z_1, \dots, z_{4n})$ *and* $\tau_n = (x_1, y_1, x_3, y_3, \dots, x_{4n-1}, y_{4n-1})(x_2, z_1, x_4, z_3, \dots, x_{4n}, z_{4n-1})(y_2, z_2, y_4, z_4, \dots, y_{4n}, z_{4n})$.

Proof. The assertions (1) and (2) is due to Theorem (3.7) and Theorem (4.7). We prove the assertion about the semi-direct structure in (2). Note that any element of G_n can be uniquely written as one of ρ^i, $\rho^i\xi_n$, $\rho^i\xi_n\rho_n$ for $0 \le i \le 4n-1$. Let $\beta_n = [\rho_n, \xi_n] \in \mathcal{D}(G_n)$. Then by an easy computation, we have $\beta_n = \rho_n^{4n-2}\xi_n\rho_n$, $\beta^2 = \rho_n^{4n-1}\xi_n$ and and $\rho_n\beta_n\rho_n^{-1} = \rho_n^{4n-1}\xi_n = \beta_n^2$. Finally we prove the assertion (3). It is easy to see that $\{\sigma_n, \tau_n\}$ satisfies the relations: $\{\sigma_n, \tau_n\} = e$, $\sigma_n^2 = \tau_n^2$, $\sigma_n^{4n} = e$. Thus we have a homomorphism $\phi : G_n \to \mathfrak{S}_{12n}$ which is defined by $\phi(\rho_n) = \sigma_n$ and $\phi(\xi_n) = \tau_n$. Let G_n' be the image. As we know $|G_n| = 12n$ and $\mathrm{ord}(\sigma_n) = 4n$, we have either $|G_n'| = 4n$ or $12n$. As $\tau_n \notin \langle\sigma_n\rangle$, we must have $|G_n'| = 12n$, which implies that $\phi : G_n \to G_n' \subset \mathfrak{S}_{12n}$ is an isomorphism. □

Remark (5.6). Composing the cyclic and Jung transformations, we can produce many different types of singularities with the same fundamental group. For example, there are at least 7 types of curves C_i, $i = 1, \dots, 7$ of degree 12 whose complements have the fundamental group G_3 as follows. (In the list, $\Sigma(C_i)$ is the singularities of C_i.)
1. $C_1 = \mathcal{C}_{1,3}(Z_4)$ and $\Sigma(C_1) = \{9B(2,3) + B(8,12)\}$. 2. $C_2 = \mathcal{C}_{2,3}(Z_4)$ and $\Sigma(C_2) = \{18B(2,3) + B(4,12)\}$. 3. $C_3 = \mathcal{C}_{3,3}(Z_4)$ and $\Sigma(C_3) = \{27B(2,3)\}$. 4. $C_4 = \mathcal{J}_3(Z_4)$ and $\Sigma(C_4) = \{3B(2,3) + B(2,3;4)\}$. 5. $C_5 = \mathcal{C}_3(\mathcal{J}_3(Z_4); D)$ where $D = \{\widetilde{x} = \alpha\}$ and $\Sigma(C_5) = \{9B(2,3) + 3B(4,8)\}$. 6. $C_6 = \mathcal{C}_2(\mathcal{J}_3(Z_4); D)$ where $D = \{\widetilde{x} = \alpha\}$ and $\Sigma(C_5) = \{6B(2,3) + B(4,28)\}$. 7. $C_7 = \mathcal{C}_3(\mathcal{J}_2(Z_4); D)$ where $D = \{\widetilde{x} = \alpha\}$ and $\Sigma(C_7) = \{9B(2,3) + B(4,24)\}$.

(C) Zariski pairs. Let C and C' be plane curves of the same degree and let $\Sigma(C) = \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ and $\Sigma(C') = \{\mathbf{a}_1', \dots, \mathbf{a}_{m'}'\}$ be the singular points of C and C' respectively. Assume that L_∞ is generic for both of them. We say that $\{C, C'\}$ is a *Zariski pair* if (1) $m = m'$ and the germ of the singularity $(C, \mathbf{a}_j)$ is topologically equivalent to $(C', \mathbf{a}_j')$ for each j and (2) there exist neighborhoods $N(C)$ and $N(C')$ of C and C' respectively so that $(N(C), C)$ and $(N(C'), C')$ are homeomorphic and (3) the pair $(\mathbf{P}^2, C)$ is not homeomorphic to the pair $(\mathbf{P}^2, C')$ ([Ba]).

The assumption (2) is not necessary if C and C' are irreducible. For our purpose, we replace (3) by one of the following:
(Z-1) $\pi_1(\mathbf{P}^2 - C) \not\cong \pi_1(\mathbf{P}^2 - C')$,
(Z-2) $\pi_1(\mathbf{C}^2 - C^a) \not\cong \pi_1(\mathbf{C}^2 - C'^a)$, where $\mathbf{C}^2 = \mathbf{P}^2 - L_\infty$ and L_∞ is generic for C and C',
(Z-3) $\mathcal{D}(\pi_1(\mathbf{P}^2 - C)) \not\cong \mathcal{D}(\pi_1(\mathbf{P}^2 - C'))$.

We say that $\{C, C'\}$ is a *strong Zariski pair* if the conditions (1), (2) and the condition (Z-1) are satisfied. Similarly we say $\{C, C'\}$ is a *strong generic affine Zariski pair* (respectively *strong Milnor pair*) if the conditions (1), (2) and the condition (Z-2) (resp. (Z-3)) are satisfied.

If C and C' are irreducible curves satisfying (1) and (2), $\{C, C'\}$ is a strong Milnor pair if and only if the fundamental groups of the respective Milnor fibers $V(C)$ and $V(C')$ are not isomorphic by Proposition (2.7). The above three conditions (Z-1)∼ (Z-3) are related by the following.

Proposition (5.7). (1) *If* $\{C, C'\}$ *is a strong Milnor pair,* $\{C, C'\}$ *is a strong Zariski pair as well as a strong generic affine Zariski pair.*
(2) *Assume that* C *and* C' *are irreducible and assume that* $\{C, C'\}$ *is a strong Zariski pair and either* $\pi_1(\mathbf{C}^2 - C^a)$ *or* $\pi_1(\mathbf{C}^2 - C'^a)$ *satisfies* (H.I.C)*-condition. Then* $\{C, C'\}$ *is a strong generic affine Zariski pair.*

Proof. The assertion (1) is immediate by Proposition (2.3). Assume that C and C' are irreducible and assume that $\pi_1(\mathbf{C}^2 - C'^a)$ satisfies (H.I.C)-condition and assume that $\phi : \pi_1(\mathbf{C}^2 - C) \cong \pi_1(\mathbf{C}^2 - C')$ is an isomorphism. Let ω, ω' be the generators of the respective kernels of the canonical homomorphisms: $\iota_\sharp : \pi_1(\mathbf{C}^2 - C) \to \pi_1(\mathbf{P}^2 - C)$ and $\iota'_\sharp : \pi_1(\mathbf{C}^2 - C'^a) \to \pi_1(\mathbf{P}^2 - C')$. As the homology class of ω is divisible by $d = \text{degree}(C)$, the homology class of $\phi(\omega)$ is also divisible by d and therefore $\iota'_\sharp(\phi(\omega)) \in \mathcal{D}(\pi_1(\mathbf{P}^2 - C')) \cap \mathcal{Z}(\pi_1(\mathbf{P}^2 - C'))$. By (H.I.C)-condition, $\phi(\omega) \in \text{Ker}(\iota'_\sharp)$ and thus $\phi(\omega) = \omega'^j$ for some $j \in \mathbf{Z}$. As $H_1(\mathbf{C}^2 - C) \cong H_1(\mathbf{C}^2 - C') \cong \mathbf{Z}$ and $[\omega] = d$, $[\omega'] = d$, we must have $j = \pm 1$. Thus ϕ induces an isomorphism of $\text{Ker}\,\iota_\sharp$ and $\text{Ker}\,\iota'_\sharp$ and therefore an isomorphism of $\pi_1(\mathbf{P}^2 - C) \cong \pi_1(\mathbf{P}^2 - C')$ by Proposition (2.3) and by Five Lemma. □

The results of §3,4 can be restated as follows.

Theorem (5.8). *Let C, C' be projective curves and let $\mathcal{C}_{n,m}(C), \mathcal{C}_{n,m}(C')$ (respectively $\mathcal{J}_n(C)$ and $\mathcal{J}_n(C')$) be the generic (n,m)-fold cyclic transforms (resp. generic Jung transform of degree n) of C and C' respectively.*
(1) Assume that $\{C, C'\}$ is a strong affine Zariski pair (respectively strong Milnor pair). Then $\{\mathcal{C}_{n,m}(C), \mathcal{C}_{n,m}(C')\}$ is a strong affine Zariski pair (resp. strong Milnor pair).
(2) Assume that $\{C, C'\}$ is a strong Zariski pair. We assume also either C or C' satisfies (H.I.C)-condition. Then $\{\mathcal{C}_{n,m}(C), \mathcal{C}_{n,m}(C')\}$ is a strong Zariski pair.

The same assertion holds for $\mathcal{J}_n(C)$ and $\mathcal{J}_n(C')$.

Proof. The assertion (1) is due to Theorem (3.7) and Theorem (4.7). The assertion (2) follows from Theorem (3.8) and Corollary (4.10). □

A well-known example is given by Zariski ([Z1]). Let Z_6 be a curve of degree 6 with 6 cusps which are on a conic and let Z'_6 be a curve of degree 6 with 6 cusps which are not on a conic. In [O6], such examples are explicitly given. It is known that $\pi_1(\mathbf{P}^2 - Z_6)$ is isomorphic to the free product $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$ and $\pi_1(\mathbf{P}^2 - Z'_6)$ is isomorphic to $\mathbf{Z}/6\mathbf{Z}$.

Example (5.9) (A new example of a Zariski pair). In (1) $\sim$ (4), we apply generic 2-covering or $(2,2)$-covering and generic Jung transform of degree 2 to the pair $\{Z_6, Z'_6\}$ to obtain three strong Zariski pairs of curves of degree 12:
(1) Take $\{\mathcal{C}_2(Z_6), \mathcal{C}_2(Z'_6)\}$. Both curves have 12 cusps ($= B(2,3)$) and one $B(6,12)$ singularity at infinity. $\pi_1(\mathbf{P}^2 - \mathcal{C}_2(Z_6))$ is a central $\mathbf{Z}/2\mathbf{Z}$-extension of $\mathbf{Z}/3\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$ and it is denoted by $G(3;2;4)$ in [O5]. $\pi_1(\mathbf{P}^2 - \mathcal{C}_2(Z'_6))$ is isomorphic to a cyclic group $\mathbf{Z}/12\mathbf{Z}$.
(2) Take $\{\mathcal{C}_{2,2}(Z_6), \mathcal{C}_{2,2}(Z'_6)\}$. They have 24 cusps. The fundamental groups are as above.
(3) Take $\{\mathcal{J}_2(Z_6), \mathcal{J}_2(Z'_6)\}$. Singularities are 6 cusps and one $B(6,18)$. The fundamental groups are as in (1).
(4) Take $\{\mathcal{C}_2(\mathcal{J}_2(Z_6)), \mathcal{C}_2(\mathcal{J}_2(Z'_6))\}$. Singularities are 12 cusps and two $B(6,6)$ singularities.

(5) We now propose a new strong Zariski pair $\{C_1, C_2\}$ of degree 12. First for C_1, we take the generic cyclic transform $\mathcal{C}_3(Z_4)$ of degree 3 of a Zariski's three cuspidal quartic. Recall that C_1 has 9 cusps and one $B(8,12)$ singularity at $\rho_\infty := [1;0;0]$. We have seen that $\pi_1(\mathbf{P}^2 - C_1)$ is G_3, a finite group of order 36. We will construct below another irreducible curve C_2 of degree 12 with 9 cusps and one $B(8,12)$ singularity at ρ_∞ such that $\pi_1(\mathbf{P}^2 - C_2) \cong G(3;2;4)$ where $G(3;2;4)$ is introduced in [O5] (see also §6) and it is a central extension of $\mathbf{Z}/3\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$ by $\mathbf{Z}/2\mathbf{Z}$.
(6) Take $\{\mathcal{C}_{3,3}(Z_4), \mathcal{C}_3(C_2; D)\}$ where $D = \{x = \alpha\}$ is generic. They are curves of degree 12 with

27 cusps. The fundamental groups $\pi_1(\mathbf{P}^2 - C_{3,3}(Z_4))$ and $\pi_1(\mathbf{P}^2 - C_3(C_2; D))$ are isomorphic to the case (5).

Construction of $\mathbf{C_2}$. Let us consider a family of affine curves $K^a(\tau) = \{(x,y) \in \mathbf{C}^2; h(y)^3 = \tau G(x)\}$ $(\tau \in \mathbf{C}^*)$ where $h(y) = 3y^4 + 4y^3 - 1$, $G(x) = -(x^2-1)^2$.

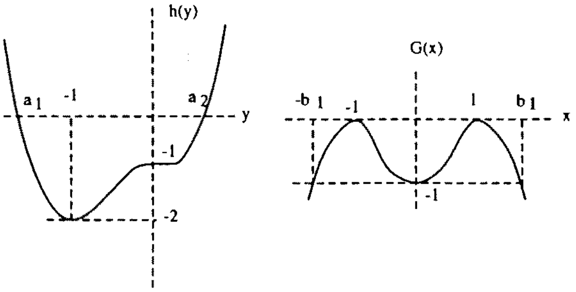

Figure (5.9.A)

Let $K(\tau)$ be the projective compactification of $K^a(\tau)$. Let $a_1, \dots, a_4$ be the solution of $h(y) = 0$. Here we assume that a_1, a_2 are real roots with $a_1 < a_2$ and $a_3 = \overline{a_4}$. By a direct computation, we see that $K(\tau)$ has 8 cusp singularities at $\{A_1, A_1', \dots, A_4, A_4'\}$ where $A_i := (1, a_i)$, $A_i' := (-1, a_i)$ for $i = 1, \dots, 4$ and a $B(8,12)$ singularity at $\rho_\infty = [1;0;0]$. Putting $\tau = 1$, $K(1)$ has one more cusp at $A_0 := (-1, 0)$. For C_2, we take $K(1)$. As $\pi_1(\mathbf{P}^2 - K(\tau)) = G(3;2;4)$ by [O5][1], $\pi_1(\mathbf{P}^2 - C_2)$ is not smaller than $G(3;2;4)$ as there exists a surjective morphism from $\pi_1(\mathbf{P}^2 - K(1))$ to $\pi_1(\mathbf{P}^2 - K(\tau)) = G(3;2;4)$. In fact, we assert that $\pi_1(\mathbf{P}^2 - C_2) = G(3;2;4)$.

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